X. Legendrian Knots

A. Preliminary Results

lemma 1

let M be a 3-manifold on which the space of contact structures isotopic to a fixed contact structure ξ is simple connected (if M has boundary only consider structures fixed at ∂M)

The classifying Legendrian knots in (M,ξ) upto isotopy is equivalent to classifying them upto contactomorphism (smoothly isotopic to the identity)

Proof:

exercise: two Legendrian knots are isotopic iff there is a contact isotopy of (M,ξ) taking one to the other

Hint: use smooth isotopy extension and the Moser method used in proof of Gray's Theorem, Th^{m} II.6

Clearly if to Legendrian knots are isotopic then they are contactomorpic by the endpoint of the ambient contact isotopy

now suppose $\phi: M \to M$ is a contactomorphism that takes one Legendrian L to the other L'

and ϕ is isotopic to the identity

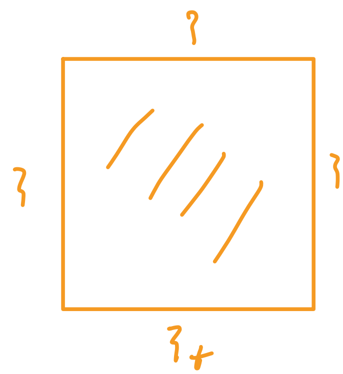
so $\exists\ \phi_t: M \to M$ with $\phi_0 = id$, $\phi_1 = \phi$

note: $\xi_t = (\phi_t)_* \xi$ is a loop of contact structures based at ξ

so by hypothesis there is a map

$$H : [0,1] \times [0,1] \longrightarrow \{\text{space of contact strs isotopic to } \xi\}$$

s.t. $H(t,0) = \xi_t$, $H(t,1) = H(0,s) = H(1,s) = \xi$



apply Grey's Th$^{\underline{m}}$ to $H(t,s)$ for $t \in [0,1]$ and s fixed

notice as s varies the diffeomorphisms constructed vary smoothly

$\therefore$ we get a map

$$F : [0,1] \times [0,1] \to \mathrm{Diffeo}(M)$$

s.t.

$F(0,s)(x) = x$

$F(t,1)(x) = x$ (since $H(t,1) = \xi$ for all t)

$F(1,s)$ are all contactomorphisms of ξ

exercise: you can choose α_t for ξ_t so that $F(t,0) = \phi_t$

$\therefore$ $F(1,s)$ is a contact isotopy from $id = F(1,1)$ to $\phi_1 = (1,0)$

Fact (Eliashberg):

the space of contact structures isotopic to ξ_{std} on S^3 or B^3 is simply connected

(Vogel):

same is true for $S^1 \times D^2$ with convex boundary having 2 dividing curves of slope n

Corollary 2:

$$\boxed{\text{the classification of Legendrian knots in } (S^3, \xi_{std}), (B^3, \xi_{std}), (S^1\times D^2, \xi_{std}) \text{ up to contactomorphism and up to isotopy is the same (for latter 2 manifolds everything is up to isotopy)}}$$

recall: any Legendrian knot L has a standard neighborhood N with convex boundary having 2 dividing curves of slope $tb(L)$

exercise:

1) any $L' \subset N$ isotopic to the core must have $tb \leq tb(L)$
2) if $tb(L') = tb(L)$ then L' is Legendrian isotopic to L

   hint: create a contactomorphism of N taking L to L' then use Corollary 2

3) any 2 standard nbhds of L (with same characteristic foliation on ∂) are isotopic

exercises imply you can study Legendrian knots by studying their standard nbhds!

note: inside N we can stabilize L

± stabilization

let $N_\pm$ be a standard neighborhood of $S_\pm(L)$

$\overline{N-N_\pm}$ is $T^2\times[0,1]$ with dividing slopes

$tb(L)-1$ and $tb(L)$

so it is a basic slice!

of course different basic slices correspond to different stabilizations

we can turn this around!

given a Legendrian L with std nbhd N

suppose N is contained in a solid torus N' and N' has convex boundary with two dividing curves of slope $tb(L)+1$

then N' is a standard nbhd of a unique Legendrian L' and L is a stabilization of L'

sign of stabilization depends on sign of the basic slice

we call L' a destabilization of L

thus if we have a bypass for ∂N along a ruling curve of slope $> tb(L)+1$, then after attaching the bypass we get a torus T' with dividing slope $tb(L)+1$ and T' bounds a solid torus N' that is a std nbhd of a destabilization of L

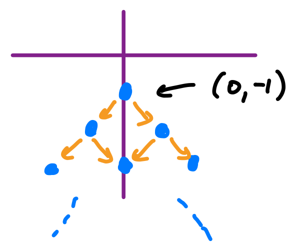
B. The unknot

Th$^{\underline{m}}$ 3:

> If U is the unknot in a tight manifold (M,ξ) then there is a unique Legendrian $L \in \mathcal{L}(U)$ with $tb(L) = -1$ (and $r(L)=0$) and all other $L' \in \mathcal{L}(U)$ are stabilizations of L

Legendrian isotopy classes of Legendrian realizations of U

note: this means the mountain range of U is



Proof:

we note the Bennequin inequality says for $L \in \mathcal{L}(U)$

$$tb(L) + |r(L)| \leq -1 \longleftarrow -\chi(D^2)$$

so we only need to consider the case when $tb(L) \leq -1$

we will show

1) any $L \in \mathcal{L}(U)$ with $tb(L) < -1$, destabilizes
2) there is a unique $L \in \mathcal{L}(U)$ with $tb(L) = -1$

the theorem clearly follows

Proof of 1):

let $L \in \mathcal{L}(U)$ with $tb(L) < -1$

let N be a standard neighborhood of L

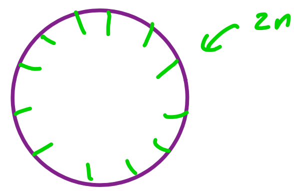
$\Gamma_{\partial N}$ is 2 curves of slope $-n$, for some $n > 1$

make ruling curves on N have slope 0

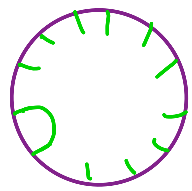
since U is the unknot $\exists$ a disk D with $\partial D =$ ruling curve note: $tw(\partial D, D) = -\frac{1}{2}(\partial D \cap \Gamma_{\partial N}) = -n$

so we can make D convex

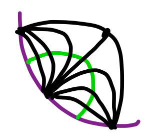
Γ_D near ∂D looks like



so we must see

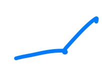


can use Giroux flexibility to find a bypass on D for ∂N



attaching bypass to ∂N gives convex torus T with dividing slope $-n+1$

T bounds N' a solid torus, as discussed above N' is a std nbhd of a Legendrian knot L' and $L = S_\pm(L')$ for some choice of sign



Proof of 2):

first assume $M = S^3$

suppose $L, L' \in \mathcal{L}(\mathcal{U})$ and $tb(L) = tb(L') = -1$

let N, N' be standard nbhds of L, L', respectively (can assume $\partial N_3 = \partial N_3'$)

set $C = \overline{S^3 - N}$ and $C' = \overline{S^3 - N'}$

these are both solid tori, naturally S°, and both have dividing slope -1

since the dividing curves are longitudinal there is a unique tight structure on S° with these boundary conditions

$\therefore \exists$ a contactomorphism $C \to C'$

and this can clearly be extended to N, N' to get a contactomorphism from $S^3 \to S^3$ taking N, N'

since $N = N'$ contains a unique Legendrian with $tb = -1$ by the discussion after Corollary 2

we can assume the contactomorphism sends L to L'

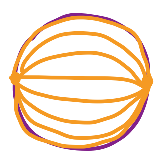
we are now done by Corollary 2

exercise: Same result holds for (B^3, ξ_{std})

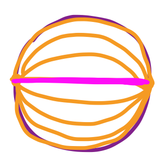
Hint: Show two Legendrian knots are isotopic in S^3 iff they are isotopic in the complement of a Darboux ball

now for L, L' in a general manifold M

we can use Giroux flexibility to show

$D_\xi =$  for L, L'

so L, L' can be isotoped into nbhd of a Legendrian arc



since we can isotop these arcs to be disjoint we can assume the disks, D, D', that L, L' bound are disjoint

let $B =$ nbhd $D \cup D' \cup$ arc joining D to D'

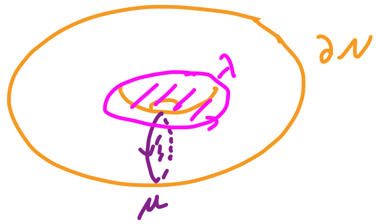
note: B is a 3-ball with a tight contact str. on it so L is isotopic to L' in B and hence in M $\blacksquare$

C. Torus Knots:

let N = nbhd of the unknot in S^3

$\mu \subset \partial N$ bound a disk in N

$\lambda \subset \partial N$ " " $\overline{S^3 - N}$



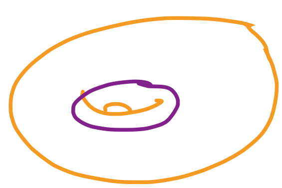
using this basis for $H_1(\partial N)$ we can represent any embedded curve γ by its homology class $p[\lambda] + q[\mu]$ for relatively prime p and q

an embedded curve γ on N realizing $p[\lambda] + q[\mu]$ is called a <u>(p,q)-torus knot</u> and is denoted $T_{p,q}$

say $T_{p,q}$ is a <u>positive torus knot</u> if $pq > 0$

is a <u>negative " </u> " $pq < 0$

<u>examples:</u> $T_{1,0}$



is the unknot

$T_{2,3}$

is the trefoil

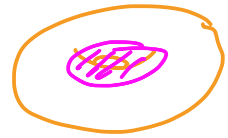
<u>exercise:</u> Show $T_{p,q}$ is isotopic to $T_{q,p}$

and $T_{-p,-q}$

exercise: Show $T_{p,q}$ has a Seifert surface of

genus $g = \frac{(p-1)(q-1)}{2}$ (same as $\chi = p+q-pq$)

Hint: take p copies of disk



q " "

they will intersect pq times on ∂N

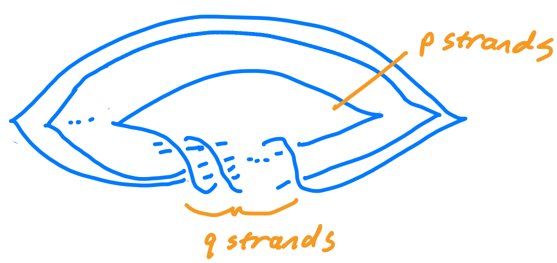
"resolve" the intersections

Th$\underline{\text{m}}$ 4:

If $T_{p,q}$ is a positive torus knot, then there is a unique Legendrian $L \in \mathcal{L}(T_{p,q})$ with $tb(L) = pq - p - q$

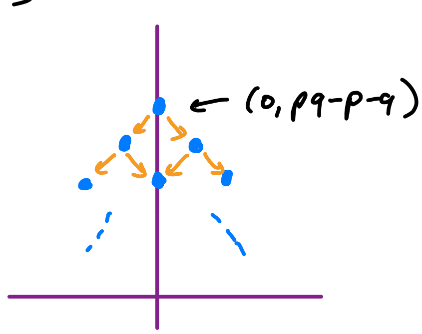
moreover, $r(L) = 0$ and any other element in $\mathcal{L}(T_{p,q})$ is a stabilization of L

L as front diagram



exercise: compute tb and r in diagram above

note: this means the mountain range of $T_{p,q}$ is



Th^{m} 5:

no loss of generality

if $T_{p,q}$ is a negative torus knot with $-q>p>0$, then

1) the maximal Thurston-Bennequin invt for knots in $\mathcal{L}(T_{p,q})$ is pq

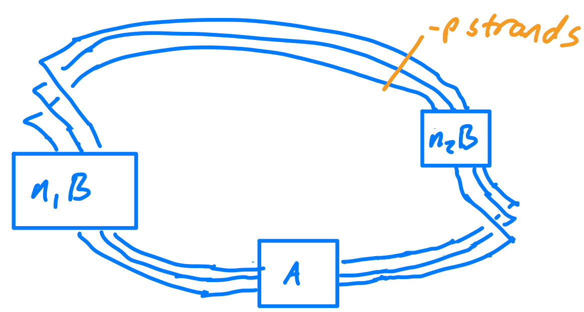
2) any knot in $\mathcal{L}(T_{p,q})$ is a stabilization of a knot with $tb=pq$

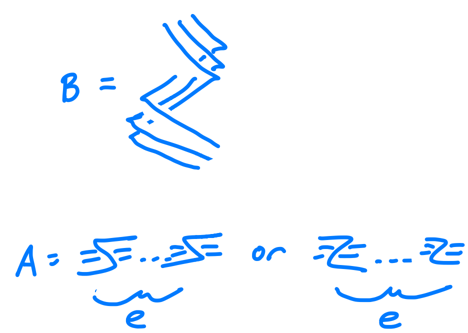
3) if $-k-1<q/p\leq -k$ then there are exactly $2k$ knots in $\mathcal{L}(T_{p,q})$ with $tb=pq$ and they are determined by there rotation numbers which are

$$\left\{ \pm(q-p+n2q) : 0\leq n\leq \frac{2(q-p)}{p} \right\}$$

4) $T_{p,q}$ is Legendrian simple (i.e. two knots in $\mathcal{L}(T_{p,q})$ are Legendrian isotopic $\Leftrightarrow$ same tb and r

if $-q=(n_1+n_2+1)p+e$ then the front diagrams for knots in $\mathcal{L}(T_{p,q})$ with $tb=pq$ are





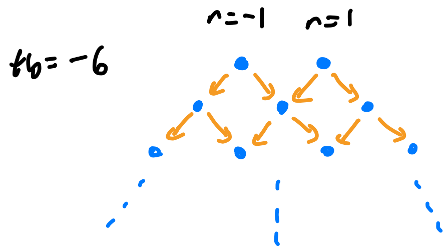
exercise: compute $tb=pq$ in examples above
compute r and show they agree with Item 3) in Th^{m}

examples:

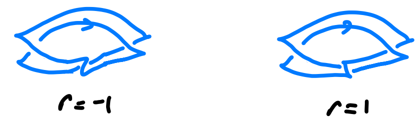
$T_{2,-3}$ $\qquad \pm(3-2-2\cdot 2n) \qquad 0\leq n< \frac{2(3-2)}{2}=1$

$\qquad\qquad \overset{\shortparallel}{\pm 1}$

so mountain range is



so all elts in $\mathcal{L}(T_{-3,2})$ are stabilizations of



more generally $T_{2,-(2n+1)}$ has $tb = -4n-2$

possible rotations are

$$\pm(2n-1-2\cdot 2\cdot n) \qquad 0 \le n < \frac{2n-1}{2}$$

$$\pm(2n-1), \pm(2n-5), \ldots, \pm(2n-4n-3)$$

eg $n=2$ $\quad \pm 3, \pm(-1)$

so $-3, -1, 1, 3$

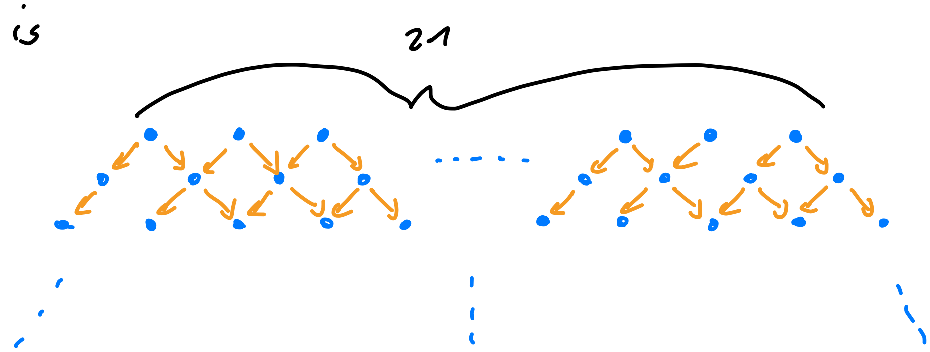
$n=3$ $\quad \pm 5, \pm 1, \pm(-3)$

so $-5, -3, -1, 1, 3, 5$

<u>exercise</u>: show rotation numbers are

$-2n+1, -2n+3, \ldots, 2n-3, 2n-1$

so mountain range is



$T_{4,-9}$ $\quad$ max $tb = -36$

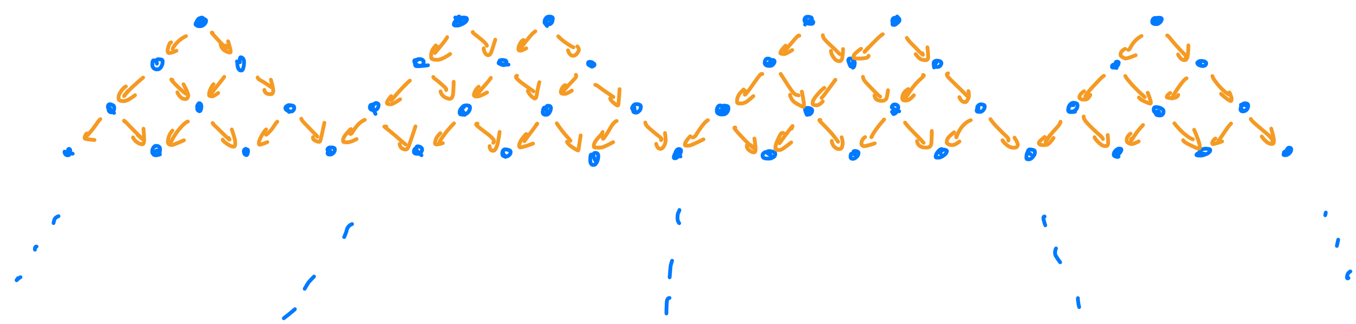
rotation numbers are $\pm(9-4-8n)$ $\qquad 0 \le n < \frac{9-4}{2} = \frac{5}{2}$

so $\pm(5-0), \pm(5-8), \pm(5-16)$

$\pm 5, \pm(-3), \pm(-11)$

$-11, -5, -3, 3, 5, 11$

so the mountain range is



exercise: Given any n, m, show there exist negative torus knots $T_{p,q}$ with mountain range having $\geq n$ "peaks" and "valleys" of depth $\geq m$

Why are positive and negative torus knots so different?

answer: slopes of convex Heegaard tori in S^3!

using the coordinates on the Heegaard torus λ, μ above (i.e. in the def^{n} of (p,q)-torus knots)

$$S^3 = S_\infty \cup S^0$$

(as we discussed when classifying contact structures on lens spaces)

we can think of S_∞ as a standard nbhd of the $tb = -1$ Legendrian unknot

so $\xi_{std}|_{S_\infty}$ is unique element in $Tight(S_\infty; -1)$

similarly $\xi_{std}|_{S^0}$ is unique element in $\mathrm{Tight}(S^0; -1)$

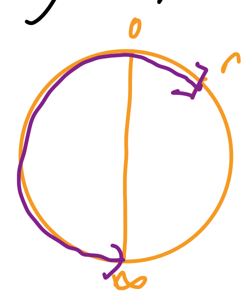
note: in S_∞ we can find convex tori with any dividing slope in $[\infty, -1]$

S^0 " " " " slope in $[-1, 0)$

so in S^3 we can find convex Heegaard tori with dividing slope any negative number!

also if T was a Heegaard torus with dividing slope $r \geq 0$ then T splits S^3 into $\tilde{S}_\infty$ and $\tilde{S}^0$ where they both have dividing slope r

thus in $\tilde{S}_\infty$ we can realize a convex torus with any dividing slope in $[\infty, r]$



in particular, there is one with dividing slope 0, a Legendrian divid on it bounds a meridional disk in $\tilde{S}^0$

$\therefore$ contact structure is overtwisted

so T does not exist!

we have shown

lemma 6:

> thinking of S^3 as $S_\infty \cup S^0$ we can find a convex Heegaard torus of slope r in $(S^3, \xi_{std}) \Leftrightarrow r<0$
> (moreover we can assume it has 2 dividing curves)

exercise:

show if S is a solid torus in a tight contact manifold (M,ξ) with core an unknot, then any convex torus T smoothly isotopic to ∂S has dividing slope in $(-\infty,0)$.
moreover, any slope in $(-\infty.0)$ can be realized as the diving slope of such a torus

classify Legendrian positive torus knots

Proof of Th$^{\underline{m}}$ 4:

we will show: 1) any element in $\mathcal{L}(T_{p,q})$ destabilizes to an element with $tb = pq-p-q$

2) there is a unique element in $\mathcal{L}(T_{p,q})$ with $tb = pq-p-q$ (and it has $r=0$)

clearly the Th$^{\underline{m}}$ follows

we first need to compair framing of $T_{p,q}$ from Seifert surface to framing coming from torus T containg $T_{p,q}$

exercise:

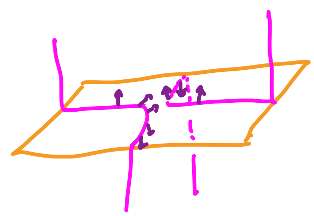
show (torus framing) $-$ (Seifert framing) $= pq$

so if $L \in \mathcal{L}(T_{p,q})$ then $tb(L) = tw(L,T) + pq$

hint: recall construction of Seifert surface using copies of disks

away from intersection points framings same

at each intersection point pick up ± 1



now the Bennequin inequality says for $L \in \mathcal{L}(T_{p,q})$

$$tb(L) \leq pq - p - q$$

so $\quad tw(L, T) \leq -p-q < 0$

$\therefore$ can make T convex without moving L

to prove 1) we assume $tb(L) < pq - p - q$

and put L on a convex torus T

exercise: if γ has slope $s \in (-\infty, 0)$

then $\gamma \cdot T_{p,q} \geq p+q$

with equality $\Leftrightarrow s = -1$

note there is a torus T' that is disjoint from, but isotopic to T such that T' convex

$|\Gamma_{T'}| = 2$

slope$(\Gamma_{T'}) = -1$

assume ruling slope of T' is q/p

let A be an annulus with one boundary a ruling curve on T' and the other L

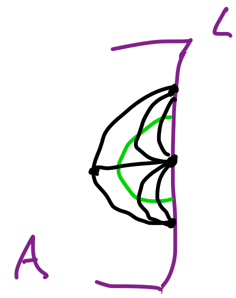
we can make A convex (why?)

note: $\Gamma_A \cap L = 2|tw(L,T)| > 2(p+q)$

$\Gamma_A \cap (\text{ruling curve}) = 2(p+q)$

so as we have done before Γ_A has a "boundary parallel" arc (parallel to L)

so we get a bypass



we can use this to directly destabilize L but arguing as in end of Section A, we can consider a standard nbhd N of L and argument above gives a bypass for ∂N along a ruling curve of slope 0 (using T framing)

so L destabilizes

to prove 2) we note if $L, L' \in \mathcal{L}(T_{p,q})$ both have $tb = pq - p - q$ then L, L' can be put on a convex torus T, T' each with 2 dividing curves of slope -1

we can also assume L, L' are ruling curves in T_1, T_1'

now T, T' bound solid tori S, S'

S, S' are std nbhds of Legendrian unknots

$\tilde{L}, \tilde{L}'$ with $tb = 1$

so Th$^{\underline{m}}$ 3 says $\tilde{L}, \tilde{L}'$ are Leg. isotopic and discussion in Section A says S is contact isotopic to S'

$\therefore$ L, L' are ruling curves on same torus

$\therefore$ isotopic through ruling curves

Proof of Th$^{\underline{m}}$ 5:

we start with

Fact: $L \in \mathcal{L}(T_{pq}) \Rightarrow tb(L) \leq pq$

$(\text{so } tw(L, T) \leq 0)$

$\therefore$ if $L \in \mathcal{L}(T_{p,q})$ then can put L on a convex torus T

if $tb(L) < pq$, then $\text{slope}(\Gamma_T) = s \neq q/p$ (or $s = q/p$ and L not a ruling curve)

$\therefore$ $\exists$ a convex torus T' disjoint from T, isotopic to T, and with $\text{slope}(\Gamma_{T'}) = q/p$ and $|\Gamma_{T'}| = 2$

(since T splits S^3 into $S_\infty \cup S^0$ and

$\xi_{std}|_{S_\infty} \in \text{Tight}(S_\infty; q/p)$

$\xi_{std}|_{S^0} \in \text{Tight}(S^0; q/p)$

first can realize all slopes in $(-\infty, s]$

second " " $[s, 0)$

and $q/p \in$ one of these intervals)

let A be an annulus with one boundary component L and other a dividing curve on T'

so as above we can find a bypass for L

and hence can destabilize L

$\therefore$ $tb(L) < pq \Rightarrow L$ destabilizes

now if $L \in \mathcal{L}(T_{p,q})$ and $tb(L) = pq$ then as above

we can put L on a convex torus with dividing

slope q/p, as a Legendrian divide

recall we are assuming $-k-1 < q/p < -k$

so there are tori T', T'' such that

T' is a convex torus with 2 dividing curves of

slope $-k$, bounding a solid torus $S' = S_\infty$

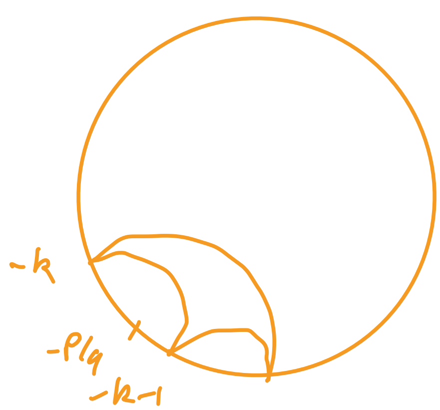
containing T

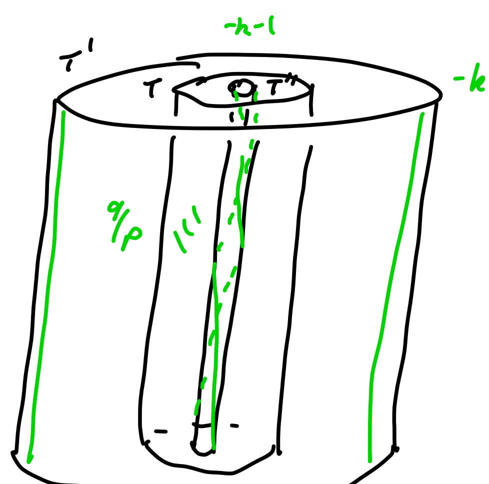
T'' is a convex torus with 2 dividing curves of

slope $-k-1$, bounding a solid torus $S'' = S_\infty$

that is contained in a solid torus $S = S_\infty$

that T bounds



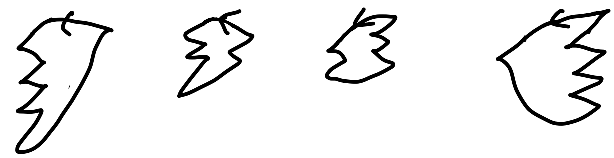


note: S' is a standard neighborhood of a

Legendrian unknot L' with $tb = -k$

there are k possibilities depending on rotation number

e.g. $k=4$



$r=-3 \quad -1 \quad 1 \quad 3$

and S'' is a standard nbhd of a Legendrian unknot L'' that is a stabilization of L'

there are 2 choices for L'': $S_\pm(L')$

Claim: L determined by L' and L''

(i.e. if $L, \tilde{L}$ has L' isotopic to $\tilde{L}'$ and L'' isotopic to $\tilde{L}''$ then L isotopic to $\tilde{L}$)

given this there are at most $2k$ $L \in \mathcal{L}(T_{p,q})$ with $tb = pq$

from exercise (front diagrams) after statement of Th$^{\underline{m}}$ 5 we know there are at least $2k$ as well and they have claimed rotation numbers

Proof of Claim:

Suppose $|\Gamma_T| = 2$

let $C' = \overline{S^3 - S'}$

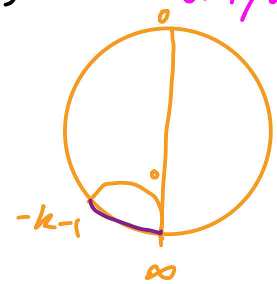
$R = \overline{S' - S''}$

$R \setminus T = R_0 \cup R_1$

note: $S^3 = S'' \cup R_0 \cup R_1 \cup C'$

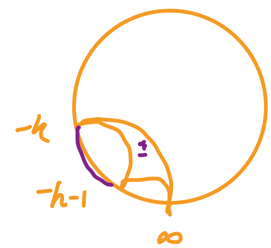
$\xi|_{S''} \in \mathrm{Tight}(S_\infty; -k-1)$

unique!



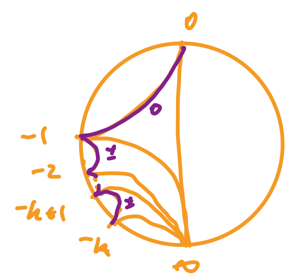
$\xi|_R \in \mathrm{Tight}_{min}(T^2 \times [0,1]; -k-1, -k)$

basic slice, so 2 possibilities
determined by ± in $L'' = S_\pm(L')$



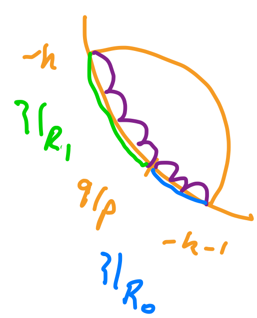
$\xi|_{C'} \in \mathrm{Tight}(S^0; -k)$

k possibilities determined by L'



finally $\xi|_{R_0}, \xi|_{R_1}$ determined by splitting $\xi|_R$
along T

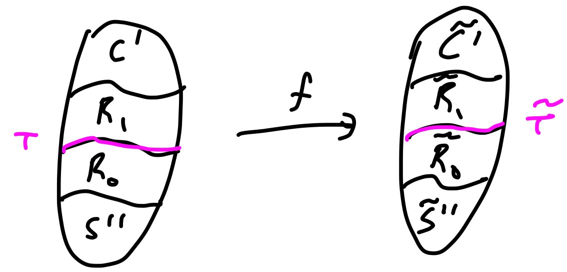
determined by $\xi|_R$
$\therefore$ by ± in $S_\pm(L') = L''$



$\therefore \exists$ contactomorphism (S^3, ξ_{std}) taking

$$S'' \longrightarrow \tilde{S}''$$
$$R_0 \longrightarrow \tilde{R}_0$$
$$R_1 \longrightarrow \tilde{R}_1$$

$C' \longrightarrow \tilde{C}'$



$\therefore$ T to $\tilde{T}$

L is Leg divide on T and

$\tilde{L}$ — — — — — — $\tilde{T}$

if f sends L to $\tilde{L}$ then we are done

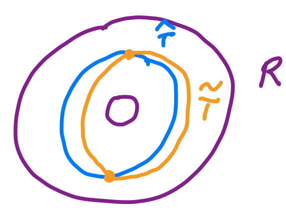
otherwise f sends L to the other Legendrian divide on $\tilde{T}$ and we are done by

exercise:

the two Legendrian divides on $\tilde{T}$ in $\tilde{R}$ are Legendrian isotopic

hint: in $\tilde{R}$ there is a torus $\hat{T}$ with $\hat{T}_\xi$ a linear foliation of slope q/p

$\tilde{T}$ is a perturbation of $\hat{T}$ and Leg. divides of $\tilde{T}$ are $\tilde{T} \cap \hat{T}$



to finish the claim we need

exercise:

suppose T has $2n$ dividing curves

in R there are tori T_1, T_2 such that

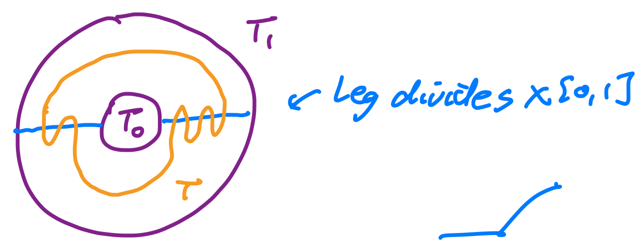
T_i are convex with two dividing curves of slope q/p

and T_0, T_1 cobound a $T^2 \times [0,1]$ containing T

the contact structure is unique on $T^2 \times [0,1]$ (proven in earlier exercise)

and the Legendrian divides on T are Legendrian isotopic to divides on T_1. so we can assume L is on a convex torus with 2 dividing curves

hint:

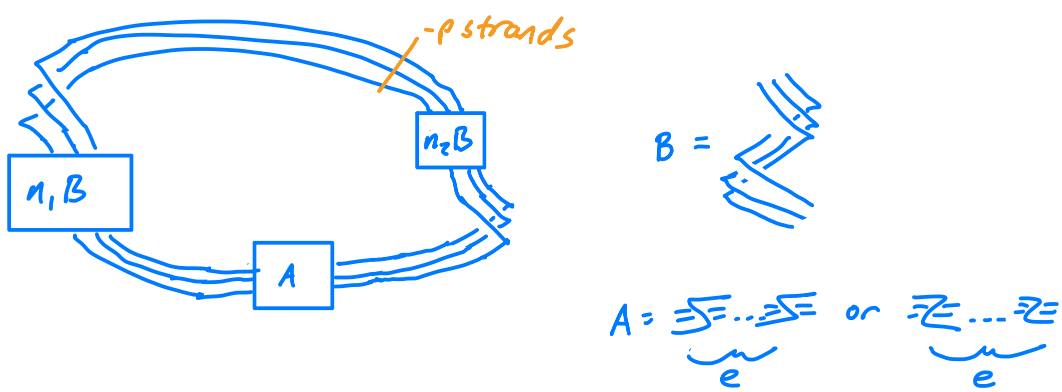


lastly we need to see if $L, L' \in \mathcal{L}(T_{p,q})$ with $tb(L) = tb(L') = pq$

and $r(L)$ is adjacent to $r(L')$ in set of rotation numbers for $tb = pq$ elts of $\mathcal{L}(T_{p,q})$ then as soon as they are stabilized so that rotation numbers are same, then they are Leg. isotopic

recall, if $-q = (n_1 + n_2 + 1)p + e$,

then the front diagrams for knots in $\mathcal{L}(T_{p,q})$ with $tb = pq$ are



exercise: show if L and L' have "adjacent" rotation numbers then either

1) n_1: for L and L' same but A's are different or

2) A's are same and n_1's differ by one

now show when L, L' differ in this way they are isotopic after stabilizing right number of times

$2^{\underline{nd}}$ way to see this

<u>exercise</u>: suppose $L, \tilde{L}$ have associated solid tori $S, S', \tilde{S}', \tilde{S}''$ as above, with $S', \tilde{S}'$ nbhd's of $L', \tilde{L}'$ and $S'', \tilde{S}''$ nbhd's of $L'', \tilde{L}''$

if $L' = \tilde{L}'$ then show first common stabilization of $L, \tilde{L}$ is the ruling curve on $\partial S' = \partial S''$

if $L' \neq \tilde{L}'$ but $L'' = \tilde{L}''$, then show the first common stabilization of $L, \tilde{L}$ is the ruling curve on $\partial S'' = \partial \tilde{S}''$

<u>Proof of Fact</u>: $pq < 0$, $L \in \mathcal{L}(T_{pq}) \Rightarrow tb(L) \leq pq$

suppose $\exists L \in \mathcal{L}(T_{p,q})$ with $tb(L) > pq$

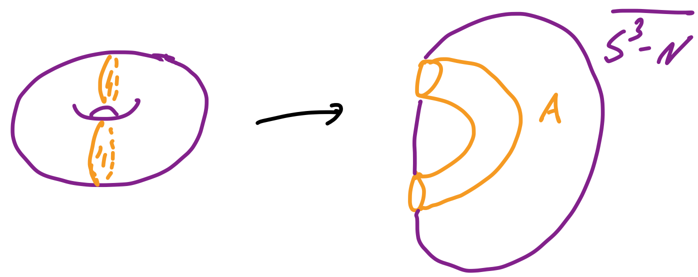
by stabilizing can assume $tb(L) = pq + 1$

now let X be Weinstein 4-mfd obtained by attaching 2-handle to B^4 along L

$\partial X = pq$-Dehn surgery on L

recall framing of L from T is pq

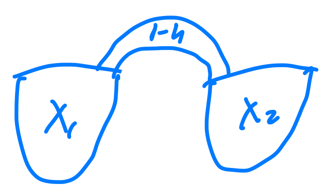
so when we do Dehn surgery we remove
a nbhd N of L from S^3
$T \cap (\overline{S^3 - N}) =$ annulus A
when we glue w/ $S^1 \times D^2$ two disks
glue to ∂A to give a sphere



$\therefore \partial X = M_1 \# M_2$

exercise: show $M_1 = L(p,q)$ and $M_2 = -L(q,p)$

Eliashberg shows if ∂X a connected sum
then $X = X_1 \cup X_2 \cup$ 1-handle



$\therefore \partial X_i = M_i$

Mayer-Vietoris $\Rightarrow X_1$ or X_2 is integral homology ball

Long exact sequence of a pair $\Rightarrow M_1$ or M_2 an integral homology sphere $\cancel{\bigcirc}$